



## Grade 11/12 Math Circles

March 1 2023

### Dynamical Systems and Fractals - Solutions

#### Exercise Solutions

##### Exercise 1

Find all of the fixed points of the function  $f(x) = x^2 - 2$ .

##### Exercise 1 Solution

Set  $f(\bar{x}) = \bar{x}$  and solve.

$$\rightarrow \bar{x}^2 - 2 = \bar{x}$$

$$\bar{x}^2 - \bar{x} - 2 = 0$$

$$(\bar{x} + 1)(\bar{x} - 2) = 0$$

This has two solutions:  $\bar{x}_1 = -1$  and  $\bar{x}_2 = 2$ , both of which must be fixed points of  $f(x) = x^2 - 2$ .

Let's check!

$$f(-1) = (-1)^2 - 2 = 1 - 2 = -1$$

$$f(2) = 2^2 - 2 = 4 - 2 = 2$$

##### Exercise 2

Given that  $f(x) = \frac{1}{x}$ , find the periodic points of period two of  $f(x)$ .

*Hint: You may want to find the fixed points of  $f(x)$  first.*

**Exercise 2 Solution**

First, find the fixed points of  $f(x)$  by solving  $f(\bar{x}) = \bar{x}$ .

$$\begin{aligned}\frac{1}{\bar{x}} &= \bar{x} \\ \bar{x}^2 &= 1\end{aligned}$$

This has solutions  $\bar{x}_1 = -1$  and  $\bar{x}_2 = 1$ . so these must be our fixed points.

Now we want to solve for the fixed points of  $f^{[2]}(x)$ .

$$\begin{aligned}f^{[2]}(x) &= f(f(x)) \\ &= \frac{1}{1/x} \\ &= x\end{aligned}$$

Setting  $f^{[2]}(\bar{x}) = \bar{x}$  gives  $\bar{x} = \bar{x}$ . This is true for all values of  $\bar{x}$ . Does this mean that all values of  $\bar{x}$  are periodic points of period two of  $f(x)$ ? Almost!

Since the points  $\bar{x}_1 = -1$  and  $\bar{x}_2 = 1$  are fixed points of  $f(x)$ , they cannot also be periodic points of period two. We also need to be careful and consider the domain of  $f(x)$ , which excludes the point  $x = 0$  (since  $f(x) = \frac{1}{x}$  is undefined when  $x = 0$ ). This means that  $x = 0$  cannot be a periodic point of  $f(x)$ . What we are left with is that all  $x \in \mathbb{R}$  except for  $x = -1, 1$ , and  $0$  are periodic points of period two of  $f(x)$ . We could write the set of periodic points of period two of  $f(x)$  as  $\{x \in \mathbb{R} | x \neq -1, 1, 0\}$ .

**Problem Set Solutions**

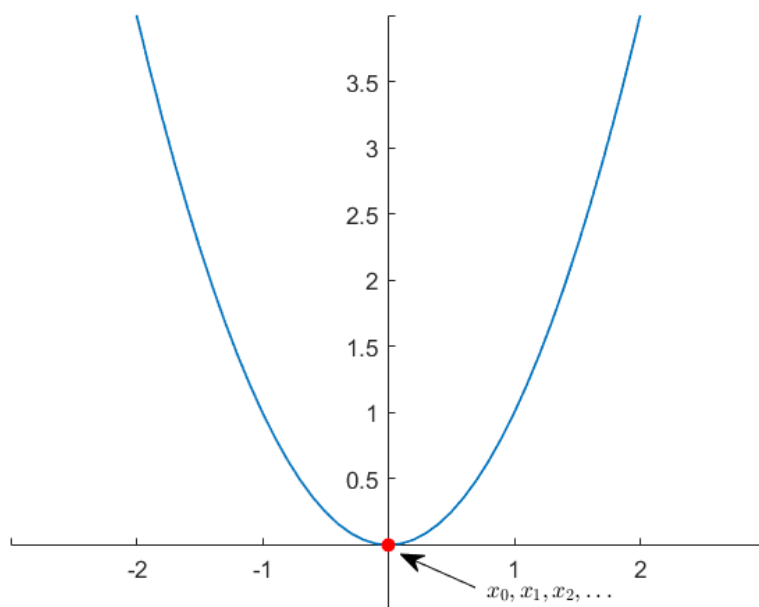
1. Consider the function  $f(x) = x^2$ . Sketch this function and plot the first few points of its orbit  $\{x_0, x_1, x_2, x_3, \dots\}$ , i.e. plot the points  $(x_0, x_1 = f(x_0))$ ,  $(x_1, x_2 = f(x_1))$ , etc..., for the starting values  $x_0 = 0, 1/2$ , and  $2$ . Describe what is happening to the orbit of  $f(x)$  for each of these starting values.



*Solution:*

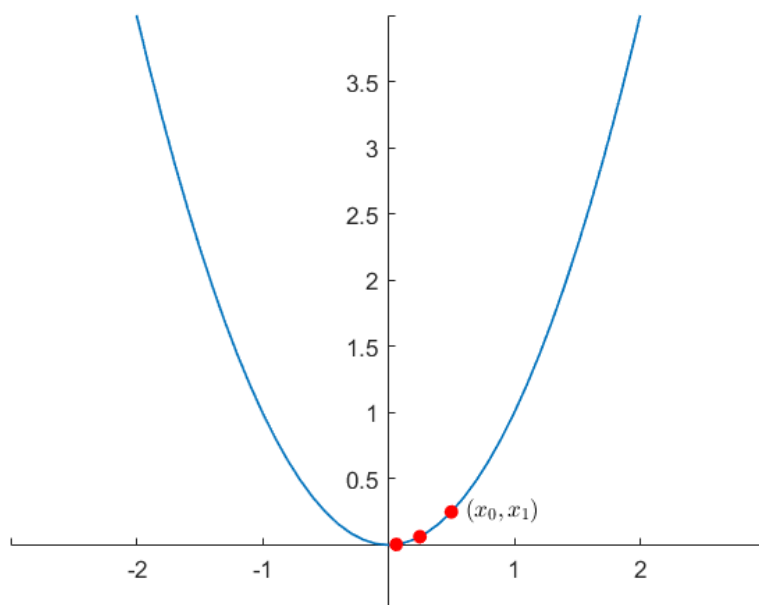
$$x_0 = 0:$$

The orbit of  $x_0 = 0$  under  $f(x) = x^2$  is:  $\{0, 0, 0, 0, \dots\}$ . The point  $x_0 = 0$  is a fixed point of  $f(x)$ .



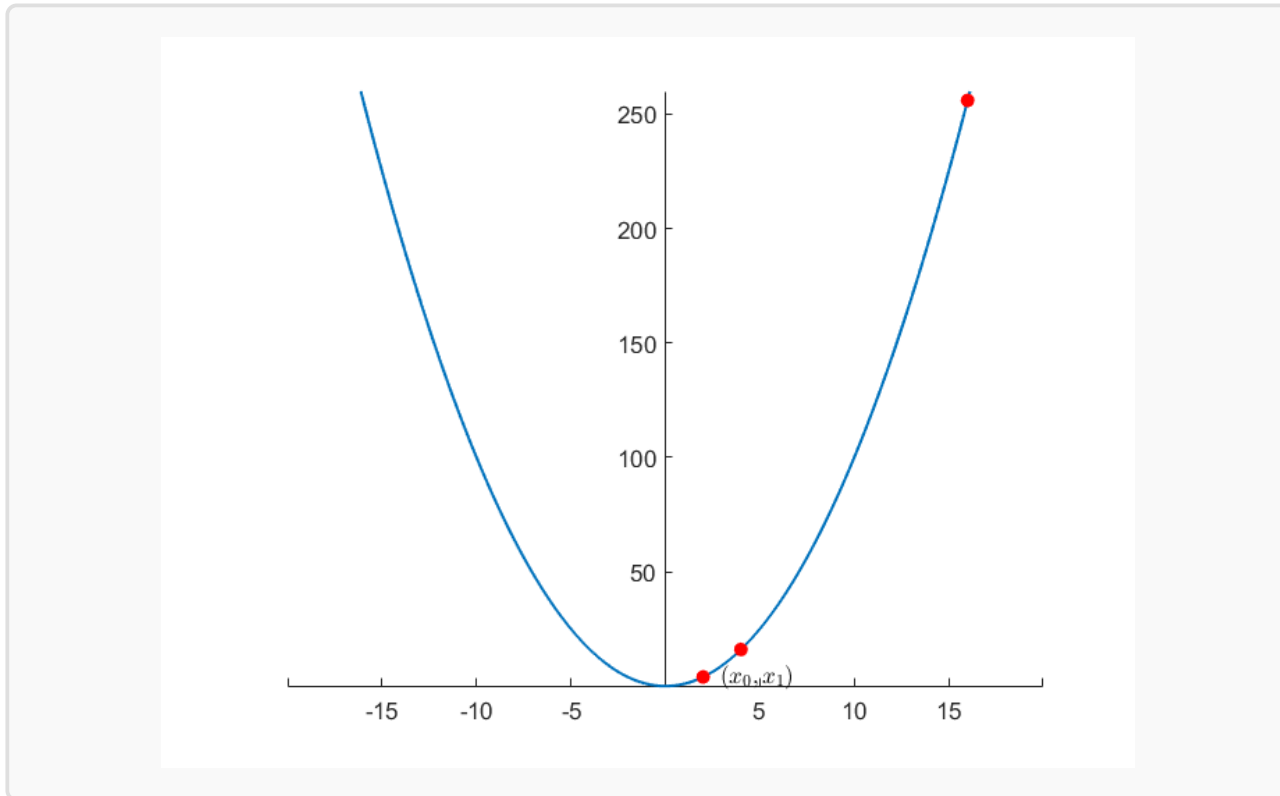
$$x_0 = 1/2:$$

The orbit of  $x_0 = 1/2$  under  $f(x) = x^2$  is:  $\{1/2, 1/4, 1/16, 1/256, \dots\}$ . The iterates of  $x_0 = 1/2$  are getting smaller in magnitude on each iteration and approaching zero.



$x_0 = 2$ :

The orbit of  $x_0 = 2$  under  $f(x) = x^2$  is:  $\{2, 4, 16, 256, \dots\}$ . The iterates of  $x_0 = 2$  are getting larger in magnitude on each iteration and approaching infinity.



2. Let  $f(x) = x^2 + 3x + 1$ . Find all of the fixed points of  $f(x)$ .

*Solution:* To find the fixed points we need to solve  $f(\bar{x}) = \bar{x}$ .

$$\bar{x}^2 + 3\bar{x} + 1 = \bar{x}$$

$$\bar{x}^2 + 2\bar{x} + 1 = 0$$

$$(\bar{x} + 1)^2 = 0$$

This has one solution,  $\bar{x} = -1$ , so  $f(x)$  has just one fixed point at  $\bar{x} = -1$ .

3. Consider the family of functions defined by  $f_c(x) = cx$  where  $c$  is a constant and  $c \neq 0$ . Determine all of the fixed points of  $f_c(x)$ .

*Hint:* You may end up with different fixed points depending on the value of  $c$ .



*Solution:* To find the fixed points of  $f_c(x)$  we solve  $f_c(\bar{x}) = \bar{x}$ , treating  $c$  as a constant.

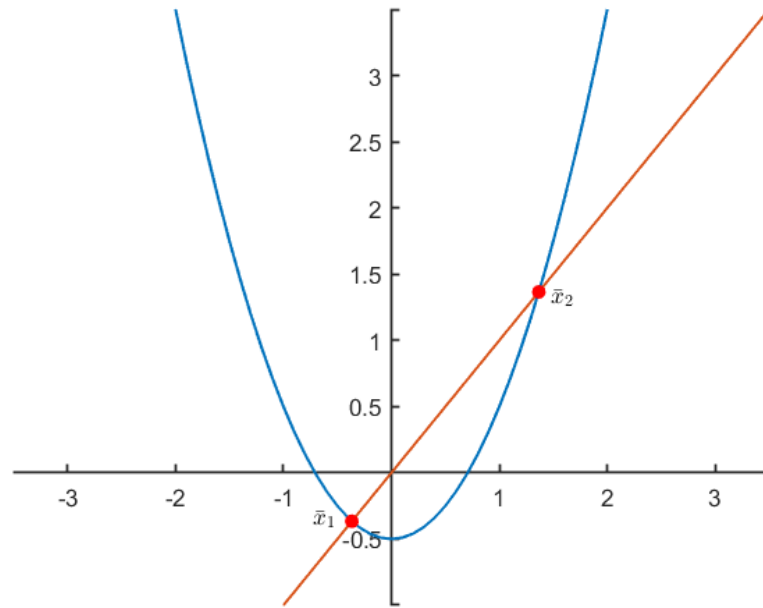
$$\begin{aligned}c\bar{x} &= \bar{x} \\(c - 1)\bar{x} &= 0\end{aligned}$$

For most values of  $c$ , this has one solution,  $\bar{x} = 0$ , however we can see that when  $c = 1$ , then all  $\bar{x} \in \mathbb{R}$  are solutions. Thus, the fixed points of  $f_c(x)$  are  $\bar{x} = 0$ ,  $c \neq 1$  and  $\bar{x} \in \mathbb{R}$ ,  $c = 1$ . Notice that when  $c = 1$ ,  $f_1(x) = x$ , for which all  $x \in \mathbb{R}$  are clearly fixed points.

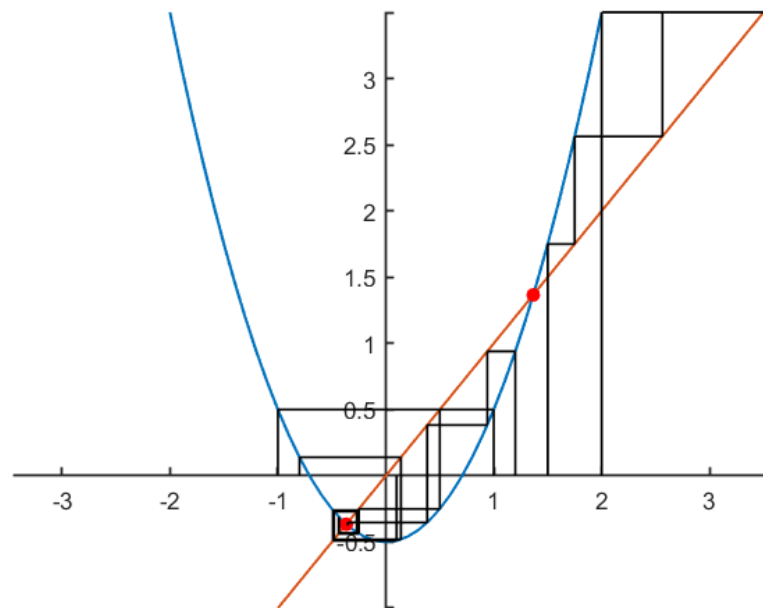
4. (a) Consider the function  $f(x) = x^2 - \frac{1}{2}$ . Sketch  $f(x)$  and  $y = x$  on the same set of axes and show graphically that  $f(x)$  has two fixed points. Label these fixed points on your sketch as  $\bar{x}_1$  and  $\bar{x}_2$  such that  $\bar{x}_1 < \bar{x}_2$ .
- (b) Use a graphical method (i.e. cobweb diagram) to help determine the behaviour of various orbits starting near both  $\bar{x}_1$  and  $\bar{x}_2$ . Use your diagram to make an educated guess as to the nature (attractive, repelling, or neither) of each fixed point.
- (c) Now consider the family of functions  $f_c(x) = x^2 + c$  where  $c$  is a constant. For what values of  $c$  do fixed points of  $f_c(x)$  exist? Some sketches of the graphs of  $f_c(x)$  for various values of  $c$  may help, but they are not necessary.

*Solution:*

- (a) From the graph, we can see that  $f(x)$  intersects the line  $y = x$  twice, and thus has two fixed points.



(b) Cobweb diagram:





From our cobweb diagram we see that the iterates of  $f(x)$  are attracted towards the fixed point  $\bar{x}_1$ , so we can guess that this is an attractive fixed point. On the other hand, the iterates of  $f(x)$  move away from the fixed point  $\bar{x}_2$ , so this is likely to be a repelling fixed point.

(c) To find the fixed points of  $f_c(x)$  we need to solve  $f_c(\bar{x}) = \bar{x}$ .

$$\begin{aligned}\bar{x}^2 + c &= \bar{x} \\ \bar{x}^2 - \bar{x} + c &= 0\end{aligned}$$

Using the quadratic formula, this has solutions  $\bar{x} = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 4c}$ . This has (real) solutions only when the argument of the square root is greater than (or equal to) zero, i.e.

$$\begin{aligned}1 - 4c &\geq 0 \\ c &\leq \frac{1}{4}.\end{aligned}$$

Thus,  $f_c(x)$  has fixed points when  $c \leq \frac{1}{4}$ . Notice that when  $c = \frac{1}{4}$ ,  $f_c(x)$  has one fixed point (only one solution to the quadratic formula) and when  $c < \frac{1}{4}$ ,  $f_c(x)$  has two fixed points (two solutions to the quadratic formula).

5. Let  $f(x) = -x^3$ . Find all fixed points and periodic points of period two of  $f(x)$ .

*Solution:* First, let's find any fixed points by solving  $f(\bar{x}) = \bar{x}$ .

$$\begin{aligned}-\bar{x}^3 &= \bar{x} \\ \bar{x} + \bar{x}^3 &= 0 \\ \bar{x}(1 + \bar{x}^2) &= 0\end{aligned}$$

This has just one solution,  $\bar{x} = 0$ , so  $f(x)$  has one fixed point at  $\bar{x} = 0$ .





Now, let's find any periodic points of period two. We need to solve  $f^{[2]}(\bar{x}) = \bar{x}$ .

$$\begin{aligned}f^{[2]}(x) &= f(f(x)) \\ &= -(-x^3)^3 \\ &= x^9\end{aligned}$$

So we need to solve  $\bar{x}^9 = \bar{x}$ . Rearranging, this gives  $\bar{x}(\bar{x}^8 - 1) = 0$ , which has three solutions  $\bar{x} = 0, -1$ , and  $1$ . Since  $\bar{x} = 0$  is a fixed point, the remaining two points must form a two cycle. Thus,  $\bar{x} = -1$  and  $1$  are the points of period two of  $f(x)$ .

6. **CHALLENGE** Let  $f(x) = 1 - x^2$ . Find all fixed points and periodic points of period two of  $f(x)$ .

*Solution:* First, let's find the fixed points.

$$\begin{aligned}1 - \bar{x}^2 &= \bar{x} \\ \bar{x}^2 + \bar{x} - 1 &= 0\end{aligned}$$

Using the quadratic formula, this gives two solutions,  $\bar{x} = \frac{-1 \pm \sqrt{5}}{2}$ , which are the two fixed points of  $f(x)$ .

Next, we want to solve for any periodic points of period two. First we find  $f^{[2]}(x)$ ,

$$\begin{aligned}f^{[2]}(x) &= 1 - (1 - x^2)^2 \\ &= 1 - 1 + 2x^2 - x^4 \\ &= 2x^2 - x^4\end{aligned}$$



and then solve  $f^{[2]}(\bar{x}) = \bar{x}$ .

$$\begin{aligned}2\bar{x}^2 - \bar{x}^4 &= \bar{x} \\ \bar{x}^4 - 2\bar{x}^2 + \bar{x} &= 0 \\ \bar{x}(\bar{x}^3 - 2\bar{x} + 1) &= 0\end{aligned}$$

Factoring the cubic part of this expression could be difficult, but luckily we know that the fixed points of  $f(x)$  are also solutions to  $f^{[2]}(\bar{x}) = \bar{x}$ . This means that  $(\bar{x}^2 + \bar{x} - 1)$  must be a factor. Thus we have

$$\bar{x}(\bar{x}^2 + \bar{x} - 1)(\bar{x} - 1) = 0$$

The two new solutions (which are not fixed points of the original function) are  $\bar{x} = 0$  and 1. Thus we must have the two cycle  $\{0, 1\}$ .

7. **CHALLENGE** Consider the function  $f(x) = x + \cos(x)$ . Show that  $f(x)$  has an infinite number of fixed points.

*Solution:* To find the fixed points of  $f(x)$  we must solve

$$\begin{aligned}\bar{x} + \cos(\bar{x}) &= \bar{x} \\ \cos(\bar{x}) &= 0.\end{aligned}$$

Considering the unit circle (or a graph of  $\cos(x)$ ), we see that this is true when  $\bar{x}$  is an odd multiple of 90 degrees, i.e.  $\bar{x} = 90^\circ, 270^\circ, 450^\circ, -90^\circ, \dots$

We can write this as  $\bar{x}$  is a fixed point of  $f(x)$  when  $\bar{x} = (2k + 1) \cdot 90^\circ$  for any integer  $k$ . Since the set of all integers is an infinite set, this results in an infinite number of fixed points for  $f(x)$ .